

On hypersemigroups

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Abstract. In this paper we show the way we pass from semigroups (without order) to hypersemigroups. Moreover we show that, exactly as in semigroups, in the results of hypersemigroups based on right (left) ideals, quasi-ideals and bi-ideals, points do not play any essential role, but the sets, which shows their pointless character. The aim of writing this paper is not just to add a publication on hypersemigroups but, mainly, to publish a paper which serves as an example to show what an hypersemigroup is and give the right information concerning this structure.

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1 Introduction and prerequisites

This paper serves as an example to show how similar is the theory of semigroups (without order) with the theory of hypersemigroups (called semihypergroups as well). A semigroup (S, \cdot) is called *regular* if for every $a \in S$ there exists an element $x \in S$ such that $a = axa$. This is equivalent to saying that $a \in aSa$ for every $a \in S$ or $A \subseteq ASA$ for every $A \subseteq S$. A semigroup (S, \cdot) is called *intra-regular* [1] if for every $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$ that is if $a \in Sa^2S$ for every $a \in S$ or $A \subseteq SA^2S$ for every $A \subseteq S$. A nonempty subset A of S is called a *right* (resp. *left*) *ideal* of (S, \cdot) if $AS \subseteq A$ (resp. $SA \subseteq A$). A nonempty subset B of S is called a *bi-ideal* of S if $BSB \subseteq B$. The concept of regular rings was introduced by J.v. Neumann [8]. L. Kovács characterized the regular rings as rings satisfying the property $A \cap B = AB$ for every right ideal A and every left ideal B of S , where AB is the set of all finite sums of the form $\sum_{a_i \in A, b_i \in B} a_i b_i$ [3]. K. Iseki studied the same for semigroups, and showed that a semigroup S is regular if and only if for every right ideal A and every left ideal B of S , we have $A \cap B = AB$ [2]. A semigroup S is intra-regular if and only if for any right

ideal A and every left ideal B of S , we have $A \cap B \subseteq BA$ [6]. In addition, a semigroup S is regular, intra-regular and both regular and intra-regular if and only if for every right ideal X , every left ideal Y and every bi-ideal B of S we have $X \cap B \cap Y \subseteq XBY$, $Y \cap B \cap X \subseteq YBX$ or $X \cap B \cap Y \subseteq BXY$, respectively [4, 5]. We examine these results for hypersemigroups in an attempt to show how similar is the theory of semigroups with the theory of hypersemigroups. We tried to use sets instead of elements to show that, exactly as in semigroups, for the results on hypersemigroups based on ideals given in this paper, points do not play any essential role, but the sets which shows their pointless character. We show that the corresponding results on semigroups are also hold as application of the results of this paper. But, though the concept of hypersemigroup generalizes the concept of a semigroup, if we want to get a result on an hypersemigroup, we never work directly on the hypersemigroup. Exactly as in Gamma-semigroups, we have to examine it first for a semigroup. The present paper serves as an example to justify what we say.

For an hypersemigroup H we have two operations. One of them is the operation between the elements of H and the other between the nonempty subsets of H . If we show these two operations with the same symbol (as we have often seen in the bibliography), a great confusion erases. So we have to show these two operations by different symbols. As the operation between the elements of H is (mostly) denoted by “ \circ ”, we denote the operation between the nonempty subsets of H by the symbol “ $*$ ”. For convenience, let us give the following definitions-notations: Let H be a nonempty set and $\mathcal{P}^*(H)$ the set of all nonempty subsets of H . Any mapping

$$\circ : H \times H \rightarrow \mathcal{P}^*(H) \mid (a, b) \rightarrow a \circ b$$

is called an *hyperoperation* on H and the pair (H, \circ) is called and *hypergroupoid*. Let “ $*$ ” be the operation on $\mathcal{P}^*(H)$ (induced by the operation of H) defined by:

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \rightarrow \mathcal{P}^*(H) \mid (A, B) \rightarrow A * B := \bigcup_{(a,b) \in A \times B} (a \circ b).$$

We can easily show that the operation “ $*$ ” is well defined.

Proposition 1. *If (H, \circ) is an hypergroupoid then, for any $A, B, C, D \in \mathcal{P}^*(H)$, we have*

- (1) $A \subseteq B$ and $C \subseteq D \implies A * C \subseteq B * D$ and
(2) $A \subseteq B \implies A * C \subseteq B * C$ and $C * A \subseteq C * B$.

Definition 2. An hypergroupoid (H, \circ) is called *hypersemigroup* if

$$\{x\} * (y \circ z) = (x \circ y) * \{z\}$$

for all $x, y, z \in H$.

Since $\{x\} * (y \circ z) = \bigcup_{\substack{a \in \{x\} \\ b \in (y \circ z)}} (a \circ b) = \bigcup_{b \in (y \circ z)} (x \circ b)$ and
 $(x \circ y) * \{z\} = \bigcup_{\substack{a \in \{x \circ y\} \\ b \in \{z\}}} (a \circ b) = \bigcup_{a \in (x \circ y)} (a \circ z)$, an hypergroupoid H is an hypersemigroup if and only if $\bigcup_{b \in (y \circ z)} (x \circ b) = \bigcup_{a \in (x \circ y)} (a \circ z)$ for every $x, y, z \in H$.

In addition, we denote by $\mathcal{P}^*(H)$ the set of nonempty subsets of H and, for an hypersemigroup, we certainly have to show that the operation “ $*$ ” on $\mathcal{P}^*(H)$ satisfies the associativity relation which leads to the associativity relation on any finite sequence A_1, A_2, \dots, A_n of elements of $\mathcal{P}^*(H)$. Unless this associativity relation, in an expression of the form $A_1 * A_2 * \dots * A_n$ of elements of $\mathcal{P}^*(H)$ it is not known where to put the parenthesis, and so all the results we get are without sense.

We use the terms left (right) ideal, bi-ideal, quasi-ideal instead of left (right) hyperideal, bi-hyperideal, quasi-hyperideal and so on, and this is because in this structure there are not two kind of left ideals, for example, to distinguish them as left ideal and left hyperideal. The left ideal in this structure is that one which corresponds to the left ideal of semigroups.

2 Main results

Remark 3. If H is an hypergroupoid then, for every $x, y \in H$, we have

$$\{x\} * \{y\} = x \circ y.$$

Indeed, $\{x\} * \{y\} = \bigcup_{\substack{u \in \{x\} \\ v \in \{y\}}} (u \circ v) = x \circ y$.

Let us give first an example of an hypersemigroup.

Example 4. If (S, \cdot) is a semigroup and “ \circ ” the mapping of $S \times S$ into $\mathcal{P}^*(S)$ defined by

$$\circ : S \times S \rightarrow \mathcal{P}^*(S) \mid (x, y) \rightarrow x \circ y := \{xy\},$$

then (S, \circ) is an hypersemigroup. Indeed,

$$\begin{aligned} \{x\} * (y \circ z) &= \{x\} * \{yz\} \text{ (the operation “*” is well defined)} \\ &= x \circ (yz) \text{ (by Remark 3)} \\ &= \{x(yz)\} = \{(xy)z\} = (xy) \circ z \\ &= \{xy\} * \{z\} \text{ (by Remark 3)} \\ &= (x \circ y) * \{z\} \text{ (the operation “*” is well defined).} \end{aligned}$$

The following proposition, though clear, plays an essential role in the theory of hypersemigroups.

Proposition 5. *Let (H, \circ) be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*

- (1) $x \in A * B \iff x \in a \circ b$ for some $a \in A, b \in B$.
- (2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

Proposition 6. *Let (H, \circ) be an hypergroupoid and $A_i, B \in \mathcal{P}^*(H)$, $i \in I$. Then we have the following:*

- (1) $(\bigcup_{i \in I} A_i) * B = \bigcup_{i \in I} (A_i * B)$.
- (2) $B * (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (B * A_i)$.

Proof. (1) Let $x \in (\bigcup_{i \in I} A_i) * B$. By Proposition 5(1), we have $x \in a \circ b$ for some $a \in (\bigcup_{i \in I} A_i)$, $b \in B$. Since $a \in A_j$ for some $j \in I$ and $b \in B$, by Proposition 5(2), we have $a \circ b \subseteq A_j * B \subseteq \bigcup_{i \in I} (A_i * B)$. Let now $x \in \bigcup_{i \in I} (A_i * B)$. Then $x \in A_j * B$ for some $j \in I$. By Proposition 5(1), we have $x \in a \circ b$ for some $a \in A_j$, $b \in B$. Since $a \in A_j \subseteq \bigcup_{i \in I} A_i$ and $b \in B$, by Prop. 5(2), we have $a \circ b \subseteq (\bigcup_{i \in I} A_i) * B$. Then we get $x \in (\bigcup_{i \in I} A_i) * B$.

The proof of property (2) is similar. □

Proposition 7. *Let (H, \circ) be an hypergroupoid, $A_i, B \in \mathcal{P}^*(H)$, $i \in I$ and $\bigcap_{i \in I} A_i \neq \emptyset$. Then we have the following:*

$$(1) \left(\bigcap_{i \in I} A_i \right) * B \subseteq \bigcap_{i \in I} (A_i * B).$$

$$(2) B * \left(\bigcap_{i \in I} A_i \right) \subseteq \bigcap_{i \in I} (B * A_i).$$

Proof. Let us prove the property (1). The proof of (2) is similar.

(1) First of all, $\bigcap_{i \in I} A_i \in \mathcal{P}^*(H)$ (since $\bigcap_{i \in I} A_i \neq \emptyset$). Let $x \in \left(\bigcap_{i \in I} A_i \right) * B$. Then, by Prop. 5(1), we have $x \in a \circ b$ for some $a \in \bigcap_{i \in I} A_i$, $b \in B$. Since $a \in A_i \forall i \in I$ and $b \in B$, by Prop. 5(2), we have $a \circ b \subseteq A_i * B \forall i \in I$. Thus we get $x \in a \circ b \subseteq \bigcap_{i \in I} (A_i * B)$. \square

Proposition 8. Let (H, \circ) be an hypersemigroup and $A, B, C \in \mathcal{P}^*(H)$. Then we have

$$(A * B) * C = A * (B * C) = \bigcup_{(a,b,c) \in A \times B \times C} \left((a \circ b) * \{c\} \right).$$

Proof. In fact,

$$\begin{aligned} (A * B) * C &= \left(\bigcup_{(a,b) \in A \times B} (a \circ b) \right) * C \text{ (the operation “*” is well defined)} \\ &= \bigcup_{(a,b) \in A \times B} \left((a \circ b) * C \right) \text{ (by Proposition 6(1))} \\ &= \bigcup_{(a,b) \in A \times B} \left((a \circ b) * \bigcup_{c \in C} \{c\} \right). \end{aligned}$$

By Proposition 6(2), we have

$$(a \circ b) * \bigcup_{c \in C} \{c\} = \bigcup_{c \in C} \left((a \circ b) * \{c\} \right).$$

Thus we have

$$(A * B) * C = \bigcup_{(a,b) \in A \times B} \left(\bigcup_{c \in C} \left((a \circ b) * \{c\} \right) \right) = \bigcup_{(a,b,c) \in A \times B \times C} \left((a \circ b) * \{c\} \right).$$

On the other hand,

$$\begin{aligned} A * (B * C) &= A * \left(\bigcup_{(b,c) \in B \times C} (b \circ c) \right) = \bigcup_{(b,c) \in B \times C} \left(A * (b \circ c) \right) \\ &= \bigcup_{(b,c) \in B \times C} \left(\left(\bigcup_{a \in A} \{a\} \right) * (b \circ c) \right) \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{(b,c) \in B \times C} \left(\bigcup_{a \in A} \left(\{a\} * (b \circ c) \right) \right) \\
&= \bigcup_{(a,b,c) \in A \times B \times C} \left(\{a\} * (b \circ c) \right).
\end{aligned}$$

Since $(H, *)$ is a hypersemigroup, we have $(a \circ b) * \{c\} = \{a\} * (b \circ c)$, thus we have $(A * B) * C = A * (B * C)$. Moreover, since $b \circ c = \{b\} * \{c\}$, we have $\{a\} * (b \circ c) = \{a\} * \{b\} * \{c\}$. \square

According to Proposition 8, we write

$$(A * B) * C = A * (B * C) = A * B * C.$$

For convenience, it is no harm to prove the following proposition as well, needed in the definition of intra-regular hypersemigroups.

Proposition 9. *Let (H, \circ) be an hypersemigroup and $A, B, C, D \in \mathcal{P}^*(H)$. Then we have*

$$\begin{aligned}
(A * B * C) * D &= A * (B * C * D) = (A * B) * (C * D) \\
&= \bigcup_{(a,b,c,d) \in A \times B \times C \times D} \left(\left((a \circ b) * \{c\} \right) * \{d\} \right).
\end{aligned}$$

Proof. By Proposition 8, we have

$$(A * B * C) * D = \left((A * (B * C)) \right) * D = A * \left((B * C) * D \right) = A * (B * C * D)$$

and

$$(A * B * C) * D = \left((A * B) * C \right) * D = (A * B) * (C * D).$$

So we write

$$(A * B * C) * D = A * (B * C * D) = (A * B) * (C * D) = A * B * C * D.$$

Let us give the form of the elements of $A * B * C * D$. We have

$$\begin{aligned}
(A * B * C) * D &= \left(\bigcup_{(a,b,c) \in A \times B \times C} \left((a \circ b) * \{c\} \right) \right) * D \text{ (by Prop. 8)} \\
&= \bigcup_{(a,b,c) \in A \times B \times C} \left(\left((a \circ b) * \{c\} \right) * D \right) \text{ (by Prop. 6(1))} \\
&= \bigcup_{(a,b,c) \in A \times B \times C} \left(\left((a \circ b) * \{c\} \right) * \bigcup_{d \in D} \{d\} \right).
\end{aligned}$$

By Proposition 6(2), we have

$$\left((a \circ b) * \{c\}\right) * \bigcup_{d \in D} \{d\} = \bigcup_{d \in D} \left(\left((a \circ b) * \{c\}\right) * \{d\}\right).$$

Thus we have

$$\begin{aligned} (A * B * C) * D &= \bigcup_{(a,b,c) \in A \times B \times C} \left(\bigcup_{d \in D} \left(\left((a \circ b) * \{c\} \right) * \{d\} \right) \right) \\ &= \bigcup_{(a,b,c,d) \in A \times B \times C \times D} \left(\left((a \circ b) * \{c\} \right) * \{d\} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \left((a \circ b) * \{c\}\right) * \{d\} &= \left(\{a\} * (b \circ c)\right) * \{d\} = \{a\} * \left((b \circ c) * \{d\}\right) \\ &= \{a\} * \{b\} * \{c\} * \{d\}. \end{aligned}$$

□

By Proposition 8, using induction (exactly as in Proposition 9), for any finite family A_1, A_2, \dots, A_n of elements of $\mathcal{P}^*(H)$, we have

$$A_1 \times A_2 \times \dots \times A_n = \bigcup_{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_n} \left(\{a_1\} * \{a_2\} * \dots * \{a_n\} \right).$$

□

We are ready now to give the correct definition of regular and intra-regular hypersemigroups and give the characterizations of them which correspond to the characterization of regular and intra-regular semigroups mentioned in the introduction.

Definition 10. An hypersemigroup (H, \circ) is called *regular* if for every $a \in H$ there exists $x \in H$ such that

$$a \in (a \circ x) * \{a\}.$$

Proposition 11. Let (H, \circ) be an hypersemigroup. The following are equivalent:

- (1) H is regular.
- (2) $a \in \{a\} * H * \{a\}$ for every $a \in H$.
- (3) $A \subseteq A * H * A$ for every $A \in \mathcal{P}^*(H)$.

Proof. (1) \implies (2). Let $a \in H$. Since a is regular, there exists $x \in H$ such that $a \in (a \circ x) * \{a\}$. Since $a \in \{a\}$ and $x \in H$, by Proposition 5(2), we have $a \circ x \subseteq \{a\} * H$. Since $a \circ x \subseteq \{a\} * H$ and $\{a\} \subseteq \{a\}$, by Proposition 1, we have $(a \circ x) * \{a\} \subseteq (\{a\} * H) * \{a\}$. Thus we have $a \in \{a\} * H * \{a\}$.

(2) \implies (3). Let $A \in \mathcal{P}^*(H)$ and $a \in A$. By (2), we have $a \in \{a\} * H * \{a\}$. Since $\{a\} \subseteq A$, by Proposition 1, we have $\{a\} * H \subseteq A * H$. Since $\{a\} * H \subseteq A * H$ and $\{a\} \subseteq A$, by Proposition 1, we have $(\{a\} * H) * \{a\} \subseteq (A * H) * A$. Then we have

$$a \in \{a\} * H * \{a\} = (\{a\} * H) * \{a\} \subseteq (A * H) * A = A * H * A$$

and $a \in A * H * A$.

(3) \implies (1). Let $a \in H$. By (3), we have $\{a\} \subseteq \{a\} * H * \{a\}$. By Proposition 8, $\{a\} * H * \{a\} = \bigcup_{h \in H} ((a \circ h) * \{a\})$. Then there exists $x \in H$ such that $a \in (a \circ x) * \{a\}$, and H is regular. \square

Proposition 12. *If H is an hypergroupoid, then we have*

- (1) $H * A \subseteq H$ and $A * H \subseteq H$ for any $A \in \mathcal{P}^*(H)$.
- (2) $H * H \subseteq H$.

Proof. (1) Let $A \in \mathcal{P}^*(H)$ and $t \in H * A$. Then, by Proposition 5(1), $t \in u \circ v$ for some $u \in H, v \in A$. Since $u, v \in H$, we have $u \circ v \in \mathcal{P}^*(H)$, that is $u \circ v \subseteq H$, thus we have $t \in H$. Similarly, $A * H \subseteq H$. The property (2) is an immediate consequence of (1). \square

Definition 13. Let (H, \circ) be an hypergroupoid. A nonempty subset A of H is called a *left* (resp. *right*) ideal of H if $H * A \subseteq A$ (resp. $A * H \subseteq A$). A subset of H which is both a left and a right ideal of H is called an *ideal* (or *two-sided ideal*) of H .

For every $A \in \mathcal{P}^*(H)$, the set $A * H$ is a right ideal of H , $H * A$ is a left ideal of H and $H * A * H$ is an ideal of H . So H is a right ideal, left ideal and an ideal of H .

For any nonempty subset A of H , we denote by $R(A)$, $L(A)$ and $I(A)$ the right ideal, left ideal and the ideal of H , respectively, generated by A , that is the least with respect to the inclusion relation left, right or ideal of H containing A . Exactly as in semigroups, one can easily prove that, for any $A \in \mathcal{P}^*(H)$, we have

$$R(A) = A \cup (A * H), L(A) = A \cup (H * A), \text{ and} \\ I(A) = A \cup (H * A) \cup (A * H) \cup (H * A * H).$$

Proposition 14. *Let (H, \circ) be an hypergroupoid. If A is a left (resp. right) ideal of H , then for every $h \in H$ and every $a \in A$, we have $h \circ a \subseteq A$ (resp. $a \circ h \subseteq A$). “Conversely”, if A is a nonempty subset of H such that $h \circ a \subseteq A$ (resp. $a \circ h \subseteq A$) for every $h \in H$ and every $a \in A$, then the set A is a left (resp. right) ideal of H .*

Proof. \implies . Let A be a left ideal of H , $h \in H$ and $a \in A$. Since $h \in H$ and $a \in A$, by Proposition 5(2), we have $h \circ a \subseteq H * A$. Since A is a left ideal of H , we have $H * A \subseteq A$. Thus we have $h \circ a \subseteq A$.

\impliedby . Let A be a nonempty subset of H such that $h \circ a \subseteq A$ for every $h \in H$ and every $a \in A$. Then $H * A := \bigcup_{u \in H, v \in A} (u \circ v) \subseteq A$, so A is a left ideal of H . \square

Proposition 15. *Let H be an hypersemigroup. If A is a right ideal and B a left ideal of H , then $A \cap B \neq \emptyset$.*

Proof. Take an element $a \in A$ and an element $b \in B$ ($A, B \neq \emptyset$). Since $\{a\} \subseteq A$ and $\{b\} \subseteq B$, by Proposition 1, we have $\{a\} * \{b\} \subseteq A * B$. By Remark 3, $\{a\} * \{b\} = a \circ b$. By Proposition 5(2) and Proposition 1, we have

$$a \circ b \subseteq A * B \subseteq A * H \subseteq A \text{ and } a \circ b \subseteq A * B \subseteq H * B \subseteq B,$$

so $a \circ b \subseteq A \cap B$. Since $a \circ b \in \mathcal{P}^*(H)$, we have $(a \circ b) \neq \emptyset$, then $A \cap B \neq \emptyset$.

Theorem 16. (cf. also [7]) *An hypersemigroup H is regular if and only if for every right ideal A and every left ideal B of H , we have*

$$A \cap B = A * B \text{ (equivalently, } A \cap B \subseteq A * B \text{)}.$$

Proof. \implies . Let A be a right ideal and B a left ideal of H . Since H is regular, for the element $A \cap B$ of $\mathcal{P}^*(H)$, we have

$$\begin{aligned} A \cap B &\subseteq \left((A \cap B) * H \right) * (A \cap B) \text{ (by Prop. 11(1) } \Rightarrow \text{ (3) and Prop. 8)} \\ &\subseteq (A * H) * (A \cap B) \subseteq (A * H) * B \text{ (by Prop. 1)} \\ &= A * (H * B) \text{ (by Prop. 8)} \\ &\subseteq A * B \subseteq (A * H) \cap (H * B) \text{ (by Prop. 1)} \\ &\subseteq A \cap B. \end{aligned}$$

Thus we have $A \cap B = A * B$.

\Leftarrow . Let $A \subseteq H$. By hypothesis and Propositions 6, 8, 9, 12 and 1, we have

$$\begin{aligned}
A &\subseteq R(A) \cap L(A) = R(A) * L(A) = \left(A \cup (A * H) \right) * \left(A \cup (H * A) \right) \\
&= (A * A) \cup \left((A * H) * A \right) \cup \left(A * (H * A) \right) \cup \left((A * H) * (H * A) \right) \\
&= (A * A) \cup (A * H * A) \cup (A * H * H * A) \\
&= (A * A) \cup (A * H * A).
\end{aligned}$$

Then, by the same propositions, we have

$$\begin{aligned}
A * A &= \left((A * A) \cup (A * H * A) \right) * A = (A * A * A) \cup (A * H * A * A) \\
&\subseteq A * H * A.
\end{aligned}$$

Hence we obtain $A \subseteq A * H * A$ and, by Proposition 11(3) \Rightarrow (1), H is regular.

□

Using elements instead of sets, a nice proof of the “ \Leftarrow ” part of the above theorem is as follows:

\Leftarrow . Let $a \in H$. By hypothesis, we have

$$\begin{aligned}
a \in R(a) * L(a) &= \left(\{a\} \cup \left(\{a\} * H \right) \right) * \left(\{a\} \cup \left(H * \{a\} \right) \right) \\
&= \left(\{a\} * \{a\} \right) \cup \left(\{a\} * H * \{a\} \right) \cup \left(\{a\} * H * H * \{a\} \right) \\
&= \left(\{a\} * \{a\} \right) \cup \left(\{a\} * H * \{a\} \right).
\end{aligned}$$

If $a \in \{a\} * H * \{a\}$ then clearly, by Proposition 1(2) \Rightarrow (1), H is regular.

If $a \in \{a\} * \{a\}$, then $\{a\} \subseteq \{a\} * \{a\}$ then, by Proposition 1,

$$a \in \{a\} * \{a\} \subseteq \{a\} * \{a\} * \{a\} \subseteq \{a\} * H * \{a\},$$

so H is regular. □

Corollary 17. *A semigroup (S, \cdot) is regular if and only if for every right ideal A and every left ideal B of S , we have*

$$A \cap B = AB \text{ (equivalently, } A \cap B \subseteq AB \text{)}.$$

Proof. For the semigroup (S, \cdot) , we consider the hypersemigroup (S, \circ) given in the Example 4, where the operation “ \circ ” is defined by $x \circ y := \{xy\}$. For

this hypersemigroup, we have $A * B = AB$ for every $A, B \in \mathcal{P}^*(S)$. in fact: Let $t \in A * B$. By Proposition 5(1), $t \in a \circ b$ for some $a \in A, b \in B$. Since $a \circ b = \{ab\}$, we have $t \in \{ab\} = \{a\}\{b\} \subseteq AB$. Let now $t \in AB$. Then $t = ab$ for some $a \in A, b \in B$. By Proposition 5(2), we have $t \in \{t\} = \{ab\} = a \circ b \subseteq A * B$.

\implies . Let $(S, .)$ be regular, A a right ideal and B a left ideal of $(S, .)$. The hypersemigroup (S, \circ) is regular. Indeed: Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = (ax)a$. Then we have

$$\begin{aligned} \{a\} &= \{(ax)a\} = (ax) \circ a = \{ax\} * \{a\} \text{ (by Remark 3)} \\ &= (a \circ x) * \{a\} \text{ (the operation “*” is well defined).} \end{aligned}$$

Then $a \in (a \circ x) * \{a\}$, so (S, \circ) is regular. The set A is a right ideal of (S, \circ) . Indeed: Let $s \in S, a \in A$. Since A is a right ideal of $(S, .)$, we have $as \in A$, then $a \circ s = \{as\} \subseteq A$. Similarly, B is a left ideal of (S, \circ) . By Theorem 16, we have $A \cap B = A * B$. Since $A * B = AB$, we have $A \cap B = AB$.

\Leftarrow . Suppose $A \cap B \subseteq AB$ for any right ideal A and any left ideal B of $(S, .)$. Then for every right ideal A and every left ideal B of (S, \circ) , we have $A \cap B \subseteq A * B$. Indeed: If A is a right ideal of (S, \circ) , $a \in A$ and $s \in S$, then $as \in \{as\} = a \circ s \subseteq A$, so A is a right ideal of $(S, .)$. If B is a left ideal of (S, \circ) , then it is a left ideal of $(S, .)$ as well. By hypothesis, we have $A \cap B \subseteq AB$. Since $AB = A * B$, we have $A \cap B \subseteq A * B$. By Theorem 16, (S, \circ) is regular. Then $(S, .)$ is regular. Indeed: Let $a \in S$. Since (S, \circ) is regular, there exists $x \in S$ such that $a \in (a \circ x) * \{a\}$. By the definition of “ \circ ” and since the operation “*” is well defined, we have

$$\begin{aligned} a \in \{ax\} * \{a\} &= ax \circ a \text{ (by Remark 3)} \\ &= \{(ax)a\}. \end{aligned}$$

Then $a = axa$, and $(S, .)$ is regular.

Definition 18. Let H be an hypersemigroup. A nonempty subset B of H is called a *bi-ideal* of H if

$$B * H * B \subseteq B.$$

Equivalent Definition: $\{a\} * \{h\} * \{b\} \subseteq B$ for every $a, b \in B$ and every $h \in H$. In fact: If $B * H * B \subseteq B$, $a, b \in B$ and $h \in H$ then, by Propositions 1 and 8, we have $\{a\} * \{h\} * \{b\} \subseteq B * H * B \subseteq B$. If $\{a\} * \{h\} * \{b\} \subseteq B$ for every $a, b \in B$

and every $h \in H$ then, by Proposition 8,

$$B * H * B = \bigcup_{(x,y,z) \in B \times H \times B} (\{x\} * \{y\} * \{z\}) \subseteq B,$$

and $B * H * B \subseteq B$.

Proposition 19. *An hypersemigroup H is regular if and only if for every right ideal X , every left ideal Y and every bi-ideal B of H , we have*

$$X \cap B \cap Y \subseteq X * B * Y.$$

Proof. \implies . Let X be a right ideal, Y a left ideal and B a bi-ideal of H . By hypothesis and Proposition 11(1) \Rightarrow (3), we have

$$\begin{aligned} X \cap B \cap Y &\subseteq (X \cap B \cap Y) * H * (X \cap B \cap Y) \\ &\subseteq \left((X \cap B \cap Y) * H * (X \cap B \cap Y) \right) * H * \left((X \cap B \cap Y) * H * (X \cap B \cap Y) \right) \\ &= \left((X \cap B \cap Y) * H \right) * \left((H \cap B \cap Y) * H * (X \cap B \cap Y) \right) * \left(H * (X \cap B \cap Y) \right) \\ &\subseteq (X * H) * (B * H * B) * (H * Y) \\ &\subseteq X * B * Y. \end{aligned}$$

\Leftarrow . Let X be a right ideal and Y a left ideal of H . By Proposition 12, H is a bi-ideal of H . By hypothesis, we have

$$X \cap Y = X \cap H \cap Y \subseteq (X * H) * Y \subseteq X * Y.$$

Then, by Theorem 16, H is regular. \square

Definition 20. An hypersemigroup (H, \circ) is called *intra-regular* if, for every $a \in H$, there exist $x, y \in H$ such that

$$a \in \{x\} * \{a\} * \{a\} * \{y\}.$$

On the other hand,

$$\begin{aligned} \{x\} * \{a\} * \{a\} * \{y\} &= (x \circ a) * \{a\} * \{y\} = \{x\} * \{a\} * (a \circ y) \\ &= (x \circ a) * (a \circ y) = \{x\} * (a \circ a) * \{y\}. \end{aligned}$$

Proposition 21. *Let (H, \circ) be an hypersemigroup. The following are equivalent:*

- (1) H is intra-regular.
- (2) $a \in H * \{a\} * \{a\} * H$ for every $a \in H$.
- (3) $A \subseteq H * A * A * H$ for every $A \in \mathcal{P}^*(H)$.

Proof. (1) \implies (2). Let $a \in H$. Since H is intra-regular, there exist $x, y \in H$ such that $a \in \{x\} * \{a\} * \{a\} * \{y\}$. Since $\{x\} \subseteq H$, by Proposition 1, we have $\{x\} * \{a\} \subseteq H * \{a\}$. Since $\{a\} \subseteq \{a\}$, by Proposition 1, we get

$$\{x\} * \{a\} * \{a\} \subseteq (H * \{a\}) * \{a\} = H * \{a\} * \{a\}.$$

Since $\{y\} \subseteq H$, again by Proposition 1, we have

$$\{x\} * \{a\} * \{a\} * \{y\} \subseteq (H * \{a\} * \{a\}) * H = H * \{a\} * \{a\} * H,$$

and $a \in H * \{a\} * \{a\} * H$.

(2) \implies (3). Let $A \subseteq H$ and $a \in A$. Since $\{a\} \subseteq A$, by Proposition 1, we have $H * \{a\} \subseteq H * A$, then $H * \{a\} * \{a\} \subseteq (H * A) * A$, and $H * \{a\} * \{a\} * H \subseteq ((H * A) * A) * H = H * A * A * H$.

(3) \implies (1). Let $a \in H$. By hypothesis and Proposition 9, we have

$$\begin{aligned} \{a\} \subseteq (H * \{a\} * \{a\}) * H &= \bigcup_{(u,v,w,t) \in H \times \{a\} \times \{a\} \times H} \left(((u \circ v) * \{w\}) * \{t\} \right) \\ &= \bigcup_{(u,t) \in H \times H} \left(((u \circ a) * \{a\}) * \{t\} \right). \end{aligned}$$

Then there exist $(x, y) \in H \times H$ such that

$$\begin{aligned} a &\in ((x \circ a) * \{a\}) * \{y\} = (\{x\} * \{a\} * \{a\}) * \{y\} \\ &= \{x\} * \{a\} * \{a\} * \{y\}, \end{aligned}$$

so H is intra-regular. □

Theorem 22. (see also [7]) *Let H be an hypersemigroup. Then H is intra-regular if and only if for every right ideal A and every left ideal B of H , we have*

$$A \cap B \subseteq B * A.$$

Proof. \implies . Since H is intra-regular, by Proposition 21(1) \Rightarrow (3), we have

$$A \cap B \subseteq H * (A \cap B) * (A \cap B) * H \subseteq (H * B) * (A * H) \subseteq B * A.$$

\Leftarrow . Let $A \subseteq H$. By hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) \cap L(A) = L(A) * R(A) = \left(A \cup (H * A) \right) * \left(A \cup (A * H) \right) \\ &= (A * A) \cup (H * A * A) \cup (A * A * H) \cup (H * A * A * H). \end{aligned}$$

Then we have

$$\begin{aligned} A * A &\subseteq (A * A * A) \cup (H * A * A * A) \cup (A * A * H * A) \cup (H * A * A * H * A) \\ &\subseteq (H * A * A * H) \cup (A * A * H), \end{aligned}$$

and $H * A * A \subseteq H * A * A * H$. Then we have $A \subseteq (A * A * H) \cup (H * A * A * H)$, from which

$$A * A \subseteq (A * A * A * H) \cup (A * H * A * A * H) \subseteq H * A * A * H.$$

Then $A * A * H \subseteq H * A * A * H$, and $A \subseteq H * A * A * H$. By Proposition 21(3) \Rightarrow (1), H is intra-regular. \square

Using elements instead of sets, the proof of the “ \Leftarrow ” part of the above theorem is as follows:

\Leftarrow . Let $a \in H$. By hypothesis, we have

$$\begin{aligned} a \in L(a) * R(a) &= \left(\{a\} \cup (H * \{a\}) \right) * \left(\{a\} \cup (\{a\} * H) \right) \\ &= (\{a\} * \{a\}) \cup (H * \{a\} * \{a\}) \cup (\{a\} * \{a\} * H) \\ &\quad \cup (H * \{a\} * \{a\} * H). \end{aligned}$$

If $a \in (H * \{a\} * \{a\} * H)$ then, by Proposition 21, H is intra-regular. If $a \in \{a\} * \{a\}$ then, by Proposition 1, we have

$$\begin{aligned} a \in \{a\} &\subseteq \{a\} * \{a\} \subseteq \{a\} * \{a\} * \{a\} \subseteq \{a\} * \{a\} * \{a\} * \{a\} \\ &\subseteq H * \{a\} * \{a\} * H. \end{aligned}$$

If $a \in H * \{a\} * \{a\}$, then

$$\begin{aligned} a \in \{a\} &\subseteq H * \{a\} * \{a\} \subseteq H * (H * \{a\} * \{a\}) * \{a\} \\ &= (H * H) * \{a\} * \{a\} * \{a\} \subseteq H * \{a\} * \{a\} * H. \end{aligned}$$

If $a \in \{a\} * \{a\} * H$, in a similar way we get $a \in H * \{a\} * \{a\} * H$, and H is intra-regular. \square

Corollary 23. A semigroup $(S, .)$ is intra-regular if and only if for every right ideal A and every left ideal B of S we have $A \cap B \subseteq BA$.

Proof. As in Corollary 17, we consider the hypersemigroup (S, \circ) where $x \circ y = \{xy\}$ for every $x, y \in S$. We have already seen that A is a left (resp. right) ideal of $(S, .)$ if and only if it is a left (resp. right) ideal of (S, \circ) and that $A * B = AB$. It remains to prove that $(S, .)$ is intra-regular if and only if (S, \circ) is intra-regular: Let $a \in S$. If $(S, .)$ is intra-regular, there exist $x, y \in S$ such that $a = xa^2y$. Then we have

$$\begin{aligned} a \in \{a\} &= \{xa^2y\} = \{(xa)(ay)\} = (xa) \circ (ay) \\ &= \{xa\} * \{ay\} = (x \circ a) * (a \circ y), \end{aligned}$$

and the hypersemigroup (S, \circ) is intra-regular. If (S, \circ) is intra-regular, then there exist $x, y \in S$ such that

$$\begin{aligned} a \in (x \circ a) * (a \circ y) &= \{xa\} * \{ay\} = (xa) \circ (ay) \\ &= \{xa^2y\}, \end{aligned}$$

then $a = xa^2y$ and the semigroup $(S, .)$ is intra-regular. \square

According to Proposition 19, if $(S, .)$ is a semigroup, then for any right ideal X , any left ideal Y and any bi-ideal B of S , we have $X \cap B \cap Y \subseteq X * B * Y$. In a similar way as in Proposition 19, we can prove the following two propositions which also generalize the corresponding results of semigroups mentioned in the introduction.

Proposition 24. *An hypersemigroup H is intra-regular if and only if for every right ideal X , every left ideal Y and every bi-ideal B of H , we have*

$$X \cap B \cap Y \subseteq Y * B * X.$$

Proposition 25. *An hypersemigroup H is both regular and intra-regular if and only if for every right ideal X , every left ideal Y and every bi-ideal B of H , we have*

$$X \cap B \cap Y \subseteq B * X * Y.$$

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